The Ricardian Rent Revisited

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Abstract. The Ricardian dynamics of extension or intensification of cultivation are characterized by a continuous adaptation of activity levels to demand in spite of discontinuities in prices. As the process is more complex than described by Ricardo, a condition is necessary for its working. That condition guarantees the local and global uniqueness of long-term equilibria. If it does not hold, a more general Lemke algorithm can be defined. For a given demand, the algorithm finds all equilibria and an oddity result holds.

Key words. Dynamics, Lemke algorithm, Long-term equilibria, Rent, Ricardo.

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1 Introduction

We return to the Ricardian theory of rent, a classical piece of economic analysis, with the help of modern tools. For a given level of demand, the prices are determined by the marginal agricultural methods and the industrial methods. The rent is zero on the marginal lands (those which are partly cultivated), whereas its level on more fertile and already fully cultivated lands equalizes the overall cost of production with the marginal lands. The increase of the demand for corn does not affect prices and rents as long as no marginal land becomes fully cultivated. But the conjunction of natural increase of population and higher standards of living requires the extension of cultivation on lands of a lesser quality. When this event occurs, the price of corn rises suddenly so that a new corn land becomes economically profitable. An important feature of the dynamics sketched by Ricardo is that, in spite of spasmodic changes in prices and rents, the activity levels change continuously with demand: cultivation starts on a land which had been left fallow.

The economic literature has never fully examined the following objection to Ricardo's views: when some corn land becomes fully cultivated, the price of corn (with labour as numéraire) rises but also those of other goods, because corn enters directly or indirectly into their production. It may well occur that a rice land becomes profitable *before* any new corn land. If the price of corn was risen further, the rice land which was left fallow at the previous stage would yield a positive rent and should be fully cultivated in the next state, thus leading to a physical discontinuity between consecutive equilibria. Since this is excluded by Ricardo, the only economically admissible reaction to the shortage of corn is the cultivation of rice! The response is bizarre but may prove to be adequate if the transfer of the cultivation of rice on the new land spares corn with regard to previous methods.

This paper clarifies the notion of the Ricardian dynamics and the condition of their working. The formalization is general: it covers the case of extensive as well as intensive cultivation (intensification consists of the progressive introduction of a more productive method on the same land and is an alternative economic response to an increase in demand) and allows for non-specialization of lands and for joint production in agriculture and industry. Following Sraffa (1960), we assume that the given distribution variable is the rate of profit. In section 2 we define the notion of long-term equilibrium with scarce resources in a static framework. The model consists basically of two vector inequalities, one relative to quantities, the other to prices, with complementarity relationships. The existence of a solution to this linear complementarity problem is first established by means of the Gale-Nikaido-Debreu lemma. The idea of Ricardian dynamics invites us to define an algorithm which pays attention to the continuous evolution of the activity levels (section 3). The conditions for a working of the algorithm in a way faithful to Ricardo's conception are found and a global uniqueness result is established. When these conditions are not met, the (non-Ricardian) algorithm is of the Lemke type and we study its convergence towards an equilibrium in a finite number of steps. Two types of equilibria are identified. An exhaustivity and an oddity result on the number of equilibria are established (section 4). Some references to the economic and mathematical literature are given in section 5.

2 Long-term equilibria

2.1 Formalization

Let there be g produced goods and h types of lands, or more generally scarce resources. The total number of commodities is $n = g + h$. The ith method of production is described by an input row vector $a_i \in R^g_+$ of reproducible commodities, an input row vector $\Lambda_i \in R^h_+$ of lands, an amount of homogeneous labour $l_i \in R_+$ and an output row vector $b_i \in R^g_+$ (the methods with $b_i = 0$ are momentarily excluded because they are not considered as 'methods of production'). The industrial methods are those for which the land components are zero. Constant returns prevail. By stacking the m methods of production, a matrix representation of the productive system is $(A, \Lambda, l) \longrightarrow B$. Let $\delta \in R^g_+$ and $\overline{\Lambda} \in R^h_+$ the row vector representing respectively the final demand basket and the total amounts of lands. The unknowns are the row vector $y \in R^m_+$ of activity levels, the price (column-) vector $\nu \in R^g_+$ of the produced goods and the rent (column-) vector $\rho \in R^h_+$. Labour is chosen as numéraire. The rate of profit r is given and nonnegative. The symbols $\geqslant 0$, > 0 and $>> 0$ for vectors and matrices denote nonnegativity, semipositivity and positivity, respectively.

Definition 1 *A long-term equilibrium is a solution (y, ν, ρ) to the system (P) of inequalities with complementarity relationships:*

$$
(P) \quad \begin{array}{ll}
y(B - A) \geqslant \delta & [\nu] \\
y\Lambda \leqslant \overline{\Lambda} & [\rho] \\
(1 + r)(A\nu + \Lambda\rho + l) \leqslant B\nu & [y] \\
\quad y > 0, \nu > 0, \rho \geqslant 0 &
\end{array}
$$

These relations mean respectively that demand is met and overproduced commodities have zero prices; the scarcity constraint is met and a land which is not fully cultivated pays a zero rent; all operated methods yield the ruling rate of profit and the non-operated methods do not yield more. The formalization can be given a more compact form: let us consider the $[m \times n]$ matrices $C = (B - A, -\Lambda)$ and $G = (\frac{B}{1+r} - A, -\Lambda)$, the extended demand row vector $d = (\delta, -\overline{\Lambda}) \in R^n$ and the price-and-rent column vector $p = (\nu, \Lambda) \in R^n_+$. A long-term equilibrium is a solution to the equivalent system (P)

$$
\begin{aligned}
yC &\geqslant d \quad [p] && (1)\\
Gp &\leqslant l \quad [y] && (2)\\
y &> 0, p > 0 && (3)
\end{aligned}
$$

in which the data are the matrices C and G, with $G \leqslant C$, the $[1 \times n]$ vector d and the $[m \times 1]$ vector l. The unknowns are the semipositive $[1 \times m]$ vector y and the semipositive $[n \times 1]$ vector p. In order to ensure that the semipositivity of ν follows from that of p, we assume that labour is necessary to meet the physical requirements. Here and for the existence result below (Theorem 1), it would suffice to assume that labour is directly or indirectly necessary to produce the basket d, but assuming that the labor vector is positive is simple and avoids difficulties in the working of the algortithm. Finally, even if some components of vector $d = (\delta, -\overline{\Lambda})$ are positive, the formal analysis will lead us to examine what happens for $d \leqslant 0$.

2.2 Existence

The existence of a solution can be established independently of any dynamics.

Theorem 1 *Let $G \leqslant C$ and $d \in \Delta$. Hypotheses (H_1) and (H_2) ensure the existence of a solution to system (P) for $d \nleqslant 0$:*

(H_1) $l >> 0$,
(H_2) $d \in \Gamma = \{d; \exists y > 0 \;\; d << yG\}$

Proof. Let t be a positive scalar, $\pi = (p, y, t) > 0$ and S_ε the subset of the unit simplex of R^{n+m+1} made of the vectors π such that with $t \geq \varepsilon > 0$. As the continuous function z from S_ε into R^{n+m+1}defined by the formula $z(p, y, t) = (td - yC, -tl + Gp, yl - dp + t^{-1}y(C - G)p)$ satisfies the Walras identity $\pi z(\pi) = 0$, the Gale-Nikaido-Debreu lemma implies the existence of $(p_\varepsilon, y_\varepsilon, t_\varepsilon) \in S_\varepsilon$ such that:

$$\forall \varepsilon > 0 \;\; \forall (p, y, t) \in S_\varepsilon \;\; (t_\varepsilon d - y_\varepsilon C)p + y(-t_\varepsilon l + Gp_\varepsilon) + t(y_\varepsilon l - dp_\varepsilon) + tt_\varepsilon^{-1} y_\varepsilon (C - G)p_\varepsilon) \leqslant 0$$

Consider the inequality obtained when the last (nonnegative) term on the left-hand side of that relation is ignored. When ε tends to zero, there exists a cluster point $(p_0, y_0, t_0) \in S$ such that the same inequality holds for any $(p, y, t) \in S$, therefore

$$\begin{aligned} t_0 d - y_0 C &\leqslant 0 \\ -t_0 l + G p_0 &\leqslant 0 \\ y_0 l - d p_0 &\leqslant 0 \end{aligned}$$

$(p_0 = 0, y_0 = 0)$ is excluded by $d \nleqslant 0$ and the first inequality. $(p_0 = 0, y_0 > 0)$ is excluded by (H_1) and the third inequality, therefore $p_0 > 0$. If $t_0 = 0$, hypothesis (H_2) implies $dp_0 < yGp_0 \leqslant 0$ and a contradiction is obtained with the third inequality. Therefore t_0 is positive and $(p_1, y_1) = (p_0/t_0,\, y_0/t_0)$ is a solution to relations (1) and (2). As $y_1 = 0$ is excluded since $d \nleqslant 0$, (y_1, p_1) is a solution to system (P). ■

System (P) is closely related to another. Let $\overline{C}$, respectively $\overline{G}$, be the $[(m+n) \times n]$ matrix obtained by stacking the m rows of matrix C, respectively G, and the n rows of matrix $-I$, the opposite of the identity matrix of dimension n. Similarly, let $\bar{l}$ be the $[(m+n) \times 1]$ semipositive vector obtained by complementing the m components of l by n zeroes. These additional rows represent free disposal methods of the produced commodities or, as far as lands are concerned, the fallow methods (land as the only input, and zero output). Consider the system $(\overline{P})$:

$$\begin{aligned} \overline{y}\overline{C} &= d && (4) \\ \overline{G}p &\leqslant \bar{l} \quad [\overline{y}] && (5) \\ \overline{y} &> 0 && (6) \end{aligned}$$

For $d \nleqslant 0$, a solution to (P) generates a solution to to $(\overline{P})$: it suffices to make the free disposal method of the overproduced goods operate at positive activity levels. Conversely, let $(\overline{y}, \overline{p})$ be a solution to $(\overline{P})$. Condition (5) applied to the rows of $\overline{G}$ corresponding to matrix $-I$ implies the nonnegativity of p, and $p = 0$ is excluded by (H_1). Therefore a solution to $(\overline{P})$ generates a solution to (P). Assume now that $d << 0$: there is no solution to (P) because it would entail the inequalities $0 < yl = yGp \leq yCp = dp < 0$. On the contrary, a solution to $(\overline{P})$ is $\overline{y} = (0, -d) > 0$ and $p = 0$, and that solution is unique. In other words, the solutions to $(\overline{P})$ coincide basically with those of (P), complemented by a specific solution when $d << 0$ (the limit case when $d \leqslant 0$ with some zero components can be ignored).

A geometric interpretation of problem $(\overline{P})$ allows us to give details on the nature of its solutions in the generic case. Consider a solution to $(\overline{P})$. Flukes apart, equality (4) requires that the number of nonzero components of $\overline{y}$ is at least equal to n, whereas the n degrees of freedom in the choice of vector

p do not allow to meet more than n equalities in condition (5). It turns out that, exceptional cases apart, the number of positive components of a vector $\overline{y}$ solution to $(\overline{P})$ is equal to n. The dynamic approach assumes this equality and other non-degeneracy conditions.

3 Ricardian dynamics

3.1 Description of the algorithm

Our purpose is to define a constructive Ricardian algorithm, which starts from a solution to system $(\overline{P})$ for a certain vector $d(0)$ and transforms it step by step into a solution for another vector $d(1)$ when d follows a smooth path $d(t)$. With regard to Ricardo's ideas, we have $d(t) = (\delta(t), -\overline{\Lambda})$ where the path $\delta(t)$ represents the evolution of final demand. Finding a solution to $(\overline{P})$ amounts basically to identifying the n-set J of the positive components to $\overline{y}$, i.e.the set of operated methods. Then the calculation of $(\overline{y}, \overline{p})$ satisfying (4) and (5) is immediate. In equilibrium, conditions (4) and (5) are met simultaneouslty, but we shall consider them sucessively when the equilibrium changes. Hence the definition:

Definition 2 *An n-set J of methods sustains a vector d if equality (4) holds for a vector $\overline{y} > 0$ with its n positive components in J.*

The algorithm describes the transformations of J when vector d moves. Let us start from a long-term equilibrium $(\overline{y}, \overline{p})$ and its operated methods J for a certain demand d. The basic remark inspired by Ricardo's analysis is that a small variation $d(t)$ of d can generally be met by adapting the positive components of $\overline{y}$ in order that equality (4) continues to hold, whereas the price-and-rent vector p (or price vector, for short) is unchanged. The limit of these local adaptations is reached when a positive activity level $\overline{y}_k(t)$ vanishes at $t = t_0$ and would become negative at $t = t_0 + \varepsilon, \varepsilon > 0$. This corresponds either to the full cultivation of some land if method k is a fallow method (theory of extensive rent) or to the disappearance of method k because of its low productivity (theory of intensive rent). We intend to show the existence of a neighboring n-set J', obtained by replacing in J the outgoing row k by an incoming row k' such that condition (5) holds for some new vector p'. As it will be shown that this necessary condition defines a unique neighbouring n-set J', the condition of validity of the Ricardian dynamics is that the new n-set sustains vector $d(t_0 + \varepsilon)$.

The rows of matrices $\overline{C}$ and $\overline{G}$ corresponding to J are denoted $\overline{C}_J$ and $\overline{G}_J$. Some non-degeneracy hypotheses (ND) are set in order to avoid difficulties in the working of the algorithm:

(ND) :

(ND_1) Any n rows $\overline{C}_J$ and $\overline{G}_J$ out of $\overline{C}$ and $\overline{G}$ are independent;

(ND_2) In the vector inequality $\overline{G}p \leq \overline{l}$, not more than n equalities hold simultaneously;

(ND_3) No point of the path $d(t)$ is positively generated by less than $n-1$ rows of $\overline{C}$;

(ND_4) Only finitely many points on the path $d(t)$ are positively generated by $n-1$ rows of $\overline{C}$;

(ND_5) Let a 'cell' be the cone positively generated by n rows of $\overline{C}$. The values of t for which the path $d(t)$ belongs to the interior of a cell is an open interval of $[0,1]$.

The last three hypothesis are generally met by the path $d(t) = td(1) + (1-t)d(0)$. We still assume that $d \in \Gamma$ (hypothesis (H$_2$). The set Γ is an open convex cone defined by its outer frontier. In the economic interpretation of the Ricardian dynamics, vector $d = (\delta, -\overline{\Lambda})$ belongs to the subset $E = R^n_+ \times \left\{-\overline{\Lambda}\right\}$. Hence:

Definition 3 *Let there be a smooth path $\{d(t)\}$ in $E' = E \cap \Gamma$ and a long-term equilibrium sustaining $d(0)$ with n operated methods. The Ricardian dynamics hold along that path if there exists a path $\{y(t), p(t)\}$ of long-term equilibria sustaining $\{d(t)\}$ and for which the activity levels $y(t)$ evolve continuously.*

3.2 From a cell to the next

Let us start from a generic solution at $d(0)$, which is sustained by an n-set J and is associated with a price vector p. When t increases up from its initial value, the same n-set J sustains $d(t)$ with an unchanged price vector, until some component $y_k(t)$ of $y(t)$ vanishes at $t = t_0$ and would become negative at $t_0 + \varepsilon$. In order that the activity levels vary continuously from an equilibrium to the next, a necessary (but not sufficient) condition is that the new n-set J' be made of the $n-1$ rows of J other than k and an incoming row k'. Lemma 1 shows that the incoming row k' is uniquely defined by the constraint (5).

Definition 4 *An n-set J of rows is called a candidate if its associated price vector defined by the n equalities $\overline{g}_j p = \overline{l}_j$ $(j \in J)$ is such that $\overline{G}p \leq \overline{l}$. Neighboring candidates are candidates which differ by one row.*

A candidate defines an equilibrium for any vector d it sustains. By hypothesis (ND$_2$), the inequality $\overline{g}_j p \leqslant \overline{l}_j$ is strict for $j \notin J$.

Lemma 1 *Let there be a candidate J and an arbitrary outgoing row $k \in J$ be given, with the only restriction that no vector of the type $\sum\limits_{i \in J-\{k\}} \overline{y}_i \overline{g}_i$ belongs to the frontier of Γ. Flukes apart, there exists a unique row $k', k' \notin J$, such that $J' = J \cup \{k'\} - \{k\}$ is a candidate. Morever, the procedure is reversible: J is the n-set which succeeds to J' when k' is the outgoing row of J'.*

Proof. If a vector p' satisfying the conditions required on J' exists, the vector $\pi = p' - p$ is such that $\overline{g}_j \pi = 0$ for $j \in J - \{k\}$ and $\overline{g}_k \pi < 0$. These conditions define a nonzero vector π up to a positive factor and p' must be of the type $p(\lambda) = p + \lambda \pi$ for some $\lambda > 0$. Conversely, let us wonder if $p(\lambda)$ is the price vector of a neighboring candidate. From inequality $\overline{G}p \leqslant \bar{l}$, we know that $p \geqslant 0$. Consider first a zero component p_j of vector p. By (ND$_2$), equality $p_j = 0$ occurs if the free disposal of good j belongs to J: (i) if $j \neq k$, condition $\overline{g}_j \pi = 0$ implies $\pi_j = 0$; (ii) if $j = k$, condition $\overline{g}_k \pi < 0$ implies $\pi_j > 0$. In both cases, $p_j(\lambda)$ is nonnegative for any $\lambda > 0$. Consider now the positive components of p:

(i) For any i such that $\pi_i < 0$, the free disposal of good i is not an operated method, because $\overline{g}_j \pi \leqslant 0$ for any $j \in J$. Assume first that π admits a negative component. There exists a value $\mu > 0$ such that vector $p(\mu) = p + \mu \pi$ is semipositive with some zero component i and, then, the free disposal of good i becomes profitable.

(ii) If vector π is semipositive, $p(\lambda)$ is also semipositive for any $\lambda > 0$. If moreover $\overline{g}_i \pi > 0$ for some $i \notin J$, there exists μ such that the inequality $\overline{g}_i p < \bar{l}_i$ is transformed into an equality for $p(\mu)$.

In both cases, when λ increases up from 0, there exists a *first* value λ_0 in $]0, \mu]$ and an index $k' \notin J$ such that the inequality $\overline{g}_{k'} p < \bar{l}_{k'}$ for $\lambda = 0$ is transformed into an equality at $\lambda = \lambda_0$ (by (ND$_2$), the index for which this change occurs at λ_0 is unique). All other inequalities at $\lambda = 0$ still hold at $\lambda = \lambda_0$. The n-set J' and its associated vector $p(\lambda_0) = p'$ satisfy the conditions stated in Definition 4. On the whole, the only case in which an ingoing row is not defined occurs when the following conditions are met: $\pi > 0$, $\overline{g}_j \pi = 0$ for $j \in J - \{k\}$, and $\overline{g}_i \pi \leq 0$ for $i = k$ or $i \notin J$. Then $(\sum\limits_i \overline{y}_i \overline{g}_i)\pi \leqslant 0$ for any semipositive vector $\overline{y}$, therefore $d\pi < 0$ for any vector $d \in \Gamma$. But $\widehat{d}\pi = 0$ for a vector of the type $\widehat{d} = \sum\limits_{i \in J-\{k\}} \overline{y}_i \overline{g}_i$ As $\widehat{d}\pi$ is the upper bound of the set $\{d\pi; d \in \Gamma\}$, $\widehat{d}$ belongs to the frontier of Γ, which is excluded by assumption. This shows the existence and uniqueness property of the candidate J' which succeeds to J when k is the outgoing row.

If the initial n-set is J' and k' its outgoing row, the same construction invites us to consider first the vector π' defined by $\overline{g}_j \pi' = 0$ for $j \in J' - \{k'\}$

and $\overline{g}_{k'}\pi' < 0$. Up to a positive factor, π' is equal to $-\pi$. Next, we consider the vector $p'(\lambda) = p' + \lambda\pi' = p' - \lambda\pi$ and look for the minimum value of $\lambda > 0$ for which some inequality $\overline{g}_h p'(0) < \overline{l}_h$ is transformed into the equality $\overline{g}_h p'(\lambda) = \overline{l}_h$. By definition of λ_0, no change of this type occurs on the segment $(1-\mu)p' + \mu p$ for $\mu \in]0, 1[$, therefore the first change occurs at $\mu = 1$ and row k is the incoming row: J is the new n-set which succeeds to J'. ■

Lemma 2 *Let J and J' be defined as in Lemma 1. Then* $\det \overline{G}_J$ *and* $\det \overline{G}_{J'}$ *have opposite signs.*

Proof. The neighboring n-sets J and J' have the property: for the vector p defined by $\overline{G}_J p = \overline{l}_J$ we have $\overline{g}_{k'} p < \overline{l}_{k'}$, and for the vector p' defined by $\overline{G}_J p' = \overline{l}_J$ we have $\overline{g}_k p' < \overline{l}_k$. Consider the square matrix M made of the $n-1$ rows common to J and J', with a last row equal to $\alpha\overline{g}_k + \beta\overline{g}_{k'}$. The product of matrix M by the vector $p' - p$ has $n-1$ zero components, and its last component is $\alpha(\overline{g}_k p' - \overline{l}_k) - \beta(\overline{g}_{k'} p - \overline{l}_{k'})$. For an adequate choice of $\alpha > 0$ and $\beta > 0$, the last component is also zero. Then $0 = \det(M) = \alpha \det \overline{G}_J + \beta \det \overline{G}_{J'}$, hence the conclusion. ■

In the initial equilibrium, both conditions (4) and (5) are satisfied. Given the path $d(t)$, the outgoing row is the one for which $y_k(t)$ vanishes at $t = t_0$, so that the physical requirement (4) would no longer be met at $t_0 + \varepsilon$. The above construction has defined the incoming row as the unique row for which the value condition (5) continues to hold. It comes as no surprise that the new candidate does not necessarily meet condition (4) at $t_0 + \varepsilon$: then the Ricardian dynamics fail. Lemma 3 examines when the substitution of a method for another helps to solve the physical side of the problem:

Lemma 3 *Let $J = supp(\overline{y})$ be an n-set which sustains $d(t)$ for t slightly smaller than t_0, but $\overline{y}_k(t_0) = 0$. Let J^* be the new n-set obtained by replacing row $k \in J$ by an arbitrary row $k^* \notin J$. Then J^* sustains $d = d(t)$ for t slightly greater than t_0 if and only if* $\det(\overline{C}_J)$ *and* $\det(\overline{C}_{J^*})$ *have opposite signs. If the signs are identical, J^* sustains $d(t)$ for t slightly smaller than t_0.*

Proof. For t slightly smaller than t_0, equality $d(t) = \overline{y}(t)\overline{C}$ holds with $\overline{y}_J(t) >> 0$. The component $\overline{y}_k(t)$ vanishes at $t = t_0$ and becomes negative at $t = t_0 + \varepsilon, \varepsilon > 0$. When row k is replaced by k^*, the vector decomposition $d(t) = \overline{z}(t)\overline{C}_{J^*}$ gives the formal components of $d(t)$ for J^*. At $t = t_0$, we have $\overline{y}_k(t_0) = \overline{z}_{k^*}(t_0) = 0$ and the two decompositions coincide. Therefore, for $i \in J^* - \{k^*\} = J - \{k\}$, the component $\overline{z}_i(t)$, which is equal to $\overline{y}_i(t_0) > 0$ at $t = t_0$, remains positive in a neighborhood of t_0. The component $\overline{z}_{k^*}(t)$ is close to zero in a neighborhood of t_0. From the equality $\det(\overline{c}_1, ..., \overline{c}_{k-1}, d(t), \overline{c}_{k+1}, ..., \overline{c}_n) = \overline{y}_k(t) \det(\overline{c}_1, ..., \overline{c}_{k-1}, \overline{c}_k, \overline{c}_{k+1}, ..., \overline{c_n}) =$

$\overline{y}_k(t)\det(\overline{C}_J)$ and the analogous relationship for J^*, it follows that $\overline{y}_k(t)\det(\overline{C}_J) = \overline{z}_{k*}(t)\det(\overline{C}_{J^*})$, therefore the sign of $\overline{z}_{k*}(t)$ is determined by those of $\overline{y}_k(t)$ and $\det(\overline{C}_J)\det(\overline{C}_{J^*})$. Hence the conclusions. ∎

Definition 5 *An n-set J is called white if* $\det(\overline{C}_J)\det(\overline{G}_J) > 0$*, black if* $\det(\overline{C}_J)\det(\overline{G}_J) < 0$*.*

Theorem 2 *(Local Ricardian dynamics) Let* $l_0 >> 0$*. Consider a smooth path* $\{d(t)\} \subset E'$ *and a solution sustaining* $d(0)$*. The Ricardian dynamics hold from a cell to the next if and only if the old and the new n-set of operated methods have the same color.*

Proof. Consider a solution at $d(t)$ sustained by a candidate J. When some activity level $\overline{y}_k(t)$, $k \in J$ vanishes at $t = t_0$, a unique neighboring candidate J' is defined by Lemma 1. According to Lemmas 1 and 2, it sustains an equilibrium at $d(t_0 + \varepsilon)$ if and only if $\det(\overline{C}_J)\det(\overline{G}_J)$ and $\det(\overline{C}_{J'})\det(\overline{G}_{J'})$ have the same sign. ∎

Theorem 3 *(Global Ricardian dynamics) Let* $l_0 >> 0$*. The Ricardian dynamics between any two demand baskets work if all candidates are white. Then the equilibrium reached for a given* $d \in \Gamma$ *is unique and independent of the demand path.*

Proof. If all candidates are white, a solution at $d = d(0)$ can be transferred to a solution at any $d = d(1)$ by means of succesive transformations along a smooth path. Moreover, two distinct solutions at $d(0)$ do not merge along the trajectory, because the reversibility property (Lemma 1) would then be violated, as a candidate would have two successors. Therefore the uniqueness property holds for any d provided that it holds for some $d(0)$. This is the case for $d(0) << 0$, when the only solution is made of the free disposal methods. ∎

Clearly enough, if the rate of profit is zero (golden rule), we have $G = C$ and all candidates are white.

4 The general algorithm

Suppose alternatively that, along some path, a black candidate J' follows a white candidate J. Even if the Ricardian dynamics stop, a non-Ricardian algorithm of the Lemke type can be defined: according to Lemma 3, the new candidate sustains $d(t_0 - \varepsilon)$. We examine the properties of that algorithm. Note first that a white candidate corresponds to increasing values of t, a black candidate to decreasing values, and a change of color between successive candidates to a change in the direction of t.

Let us draw an oriented smooth path from $d(0) << 0$ to $d(1) \in \Gamma$. When the algorithm goes in the right direction (t inceases), a white candidate is found, otherwise it is black. The Lemke argument holds: the algorithm never comes back to its initial position, because $d(0)$ is not sustained by a black candidate. It never comes back either to a candidate which has already been found, because each cell has exactly two 'doors' along the path (hypothesis (ND_5) is used here), and a contradiction is obtained when one contemplates the first cell in which a return would occur. The conclusion is that, eventually, the algorithm reaches $d(1)$. The procedure shows the existence of a white solution at $d(1) \in \Gamma$ and finds an odd number of solutions at any point d on the path, though not all solutions at d are necessarily met. Is there a way to find *all* equilibria at d and to establish an oddity result?

Let us build a smooth path from $d(0) << 0$ to $d(0.5) = d$, then return to the negative orthant at $d(1) << 0$ by another path. The loop starting from $d(0)$ and ending at $d(1)$ in the negative orthant is called smooth if hypothesis (ND_5) still holds, with the exception of the cell made of the negative orthant to which $d(t)$ belongs when t is close to 0 or to 1.

Theorem 4 *Flukes apart, all solutions at* $d \in \Gamma$ *are found by the Lemke algorithm along a smooth loop from* $d(0) << 0$ *to* $d(1) << 0$*.*

Proof. Let us first start from a given solution sustaining d and increase the value of t. The algorithm defines successive candidates and may return to the value $t = 0.5$ (in which case another solution sustaining d is found) or smaller values, but it never comes back to the same candidate. Therefore it reaches either $d(0)$ or $d(1)$ ultimately. Combined with the reversibility property, this shows that the initial solution at d is reached by starting from $d(0)$ or, if not, from $d(1)$.

Let us now start from $d(0)$. Then the algorithm only stops when it reaches $d(1)$. The sequence of successive candidates is unique and has the reversibility property. Therefore the given initial solution at d belongs to that sequence. ■

Theorem 5 *Flukes apart, for any* $d \in \Gamma$*, the number of white solutions exceeds that of black solutions by one.*

Proof. On a smooth loop, the algorithm reaches $q+1$ times the value $t = 0.5$ for increasing values of t (white solutions) and q times for decreasing values of t (black solutions). ■

Checking the smoothness of a loop may be uneasy. Theorem 4 still holds if the loop $\{d(t)\}$ has the less demanding property that it crosses at most once any facet of the cell $\left\{\sum_{i\in J}\overline{y}_i\overline{c}_i; \overline{y}_i > 0\right\}$, even if it enters several times into that cell. Indeed, let J be a candidate sustaining $d(t)$ for $t \in [t_{entry}, t_{exit}]$ and $\bar{t} = 0.5(t_{entry} + t_{exit})$. For a given path, the knowledge of the pair $(J,\bar{t})$ allows us to identify t_{entry} and t_{exit} and, therefrom, the two neighbouring pairs $(J', \overline{t'})$ and $(J'', \overline{t''})$ when the path is followed in one or the other direction. When a pair $(J,\bar{t})$ has been reached once, the algorithm does not return to it a second time because a contradiction would be obtained by considering the first pair for which the phenomenon occurs.

5 References to the literature

Sraffa's (1960) formalization initiated the modern studies on the theory of Ricardian rent. Many papers have been devoted since to the study of extensive cultivation or intensive cultivation proper. By contrast, Salvadori (1986) wrote down the general system (P) and used results on the bimatrix problem to assert the existence of a long-term equilibrium. The possible multiplicity of solutions (D'Agata, 1983) suggested that Ricardo's intuition on the extension of cultivation was not entirely right but the origin of the difficulty was not identified. Erreygers (1990, 1995) introduced the notion of color and stated the uniqueness condition seen in Theorem 3 (the passage from local to global uniqueness is more or less admitted). Bidard and Erreygers (1998) obtained an oddity result for multiple-product systems without scarce resources by showing that the difference between the numbers of white and black solutions is invariant when the original methods of production are distorted. Bidard (2010) studied the Ricardian dynamics for intensive cultivation proper, a case in which adequate hypotheses allow to avoid any calculation .

The replacement of one outgoing row by one incoming row, as in the simplex method for linear programming or the constructive proof of the Sperner lemma, is used to solve linear complementarity problems (Cottle, Pang and Stone, 1992) by means of the Lemke algorithm (Lemke and Howson, 1964; Lemke, 1965). With regard to that branch of the literature, Theorem 3 states a uniqueness result and Theorem 4 establishes an exhaustivity property.

While post-Sraffians studies and computational methods have privileged the discrete case, another part of the economic literature inspired by the general equilibrium theory assumes a continuum of methods with differentiability properties and states oddity results, based on the oriented index theory, analogous to Theorem 5 (Debreu, 1970; Dierker, 1972).

6 Conclusion

The Ricardian dynamics of extension or intensification of cultivation explain the existence of jumps in the evolution of prices and rents but impose continuous adaptation of activity levels to demand: a new agricultural or industrial method is introduced at a low activity level. They have been considered as logically sound, even if Sraffa rejected any *a priori* classification of lands in terms of fertility, as the order of cultivation depends on distribution. Unexpectedingly enough, when an equilibrium meets a limit because it cannot satisfy demand, the preliminary examination of the value side of the problem is decisive because it determines the unique incoming method which can possibly belong to the new equilibrium. The Ricardian dynamics work if and only if that method also meets the new physical requirements, a condition which admits a simple algebraic expression.

From a formal point of view, the Ricardian dynamics can be compared with a Lemke algorithm which would impose the profitability constraint at each step but would go on even if demand is not immediately satisfied and would stop when it is met. The extended applications of the linear complementarity problem and the Lemke algorithm let us expect that the uniqueness, oddity and exhaustivity results may find other applications in economics and game theory.

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